Teaching graph theory from a STEM perspective: selected examples of interdisciplinary problems

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Abstract

Graph Theory is known as a difficult topic at the intersection of mathematics, data mining and computer science. Although graphs are theoretical constructs studying arbitrarily complex relationships between objects, their interdisciplinary real world applications are numerous. In addition, students are becoming more familiar with computer technologies, and nowadays there is a wide range of free software packages and technologies allowing to bridge the gap between theory and practice in the context of graph theory. The first objective of this paper is to propose some ideas and use cases of how to make teaching graph theory more understandable and interesting, and the second is to highlight its relationships with other STEM disciplines, including algebra, statistical analysis, big data and software development, adhering to the concept of context-based learning. Our idea is to show students different examples of how certain notions studied in theory can be represented as a graph and visualized using simple computer programs, and also to encourage them to perform their own experiments. As examples, we consider the graph of numbers and their prime factors, given the upper bound; the visualization of a permutation group; simple random graphs and networks of financial time series. These examples were given to a small group of graduate students in computer science as homework projects, but we believe that they can be adapted for a wider audience. Furthermore, we discuss why these graphs are interesting for educational purposes and how to stimulate students' imagination and creativity in terms of context-based and project-based learning. The examples given in this paper are formalized from a mathematical and graph-theoretical point of view, and suitable for students who have at least basic knowledge in programming and early-undergraduate level of mathematics.

1. Introduction

Digital technologies are consistently becoming more frequently used by STEM educators [36], [12], [32], and [50], as well as more accessible to students. Examples include: gamification of the learning using Virtual Reality / Augmented Reality tools [12]; custom implementations of learning supporting tools [32]; e-Learning systems like Moodle; data analytics technologies [14], are used with the aim to improve teaching. There is a change in teaching practices when online technologies are used as part of innovative teaching approaches [20]. Technology enhanced learning and blended learning inspired approaches, combining face-to-face learning with implementation of computer-based / online activities [21], are suitable for teaching graph theory [50], since students have the opportunity to do their own research and exploration using input data and algorithms. As noted in [30], "the recent emphasis on engineering practices provides an opportunity for mathematics teachers to apply STEM concepts to real-world engineering design challenges for students".

In this paper, we propose four examples of graphs which are related to different subjects in the applied mathematics and/or computer science curriculum:

- A graph of a finite number of integers and their prime factors.
- A Cayley graph of the group Sym(4) with its generators.

- A simple method for generating random graphs, related to probability, statistics and many real-world problems.
- A more advanced use case of a graph, generated on the basis of empirical financial time series. This example is related to statistics, empirical finance, big data, as well as computer programming,

We have developed these examples within the context of online-based teaching in Graph theory and applications (as a response to the difficulties following the COVID-19 pandemic), as project-based homework assignments, dedicated to our graduate students in a computer science program. However, we find them relevant and flexible enough to be taught to students across different disciplines who have programming and math knowledge, and an interest in Graph theory, as discussed in more details in Section 4 and Section 5. Most of our students have an undergraduate degree in Computer sciences, Mathematics, General engineering, or Telecommunications and they have been studying mathematics and programming, including introductory courses in popular programming languages. All of our students have initial professional experience – either internships, a gap years after obtaining an undergraduate degree, or part-time work, so improving their skills for solving non-trivial programming problems is of particular interest to them. This is why we conducted a survey to better understand how they perceive these project-based assignments, and to what extent they apply to the students' intended career paths. It should be noted that the examples of projects and technologies for use, described in the following two sections, are highly flexible in design. The assignments are given in pseudo-code, but students may be provided with different levels of details depending on their level of knowledge and experience. In particular, we allowed each student to use any programming language of their choice, but in general, a specific language can be required if all students know / have studied it during lectures and lab sessions we allocate enough time for consultations about possible difficulties with homework assignments, and encourage students to discuss their approaches and current solutions in groups. All examples are considered both from a mathematical point of view and from an implementation perspective, in a way allowing students to better understand the underlying theoretical problems. Furthermore, these examples can be extended by modifying the associated programming code. One of the goals is to inspire students to explore modelling problems from different perspectives [29], [2], [37], and [40], and to consider possible generalizations, while expanding their knowledge and appreciating the existing relationship between different subjects.

This paper is organized as follows: Section 2 discusses the applicability and extensibility of the examples introduced in Section 4, and how they are relevant from a STEM perspective. Section 3 outlines the prerequisites in terms of technologies and software, needed for the implementation and visualization; Section 4 introduces four examples of interdisciplinary problems involving graph theory, along with pseudo-code for the implementation; Section 5 describes a survey given to our students that we conducted with the purpose to identify which elements of the project assignments students find useful and applicable to their future professional careers, and what did they find difficult; Section 6 concludes.

2. Application of context-based learning and project-based learning

The idea of STEM education has been considered since the 1990s in the USA. Several decades later, STEM education standards are not well established. As stated in [45], "different STEM approaches have been used in different education context even within the same country". [51] discussed the importance of STEM education for the developing America's future scientists, technologists, engineers, and mathematicians and emphasized that "to remain competitive in a growing global economy, it is imperative that we raise student's achievement in STEM subjects". [28] showed a *STEM skills gap* among the workforce in Europe. Enhancing STEM education is guided by economic reasons in developing and emerging countries.

As practice shows, STEM education aimed to improve science and mathematics as isolated disciplines (see [51]). There exist only little integration of technology or engineering (see [24]). Moreover, [47] pointed out that “technology education students might very well do some arithmetic or recognize a scientific principle at play en route to completing a design challenge, but those design challenges are almost never conceived to purposefully teach a desired science or mathematics learning outcome”. Authors of [39] noted that “STEM education is an effort to combine some or all of the four disciplines of science, technology, engineering, and mathematics into one class, unit, or lesson that is based on connections between the subjects and real-world problems. Integrated STEM curriculum models can contain STEM content learning objectives primarily focused on one subject, but contexts can come from other STEM subjects”. [27] argued that “integrated STEM education is the approach to teaching the STEM content of two or more STEM domains, bound by STEM practices within an authentic context for the purpose of connecting these subjects to enhance student learning.” Of course, there are limits to using integrated STEM education. Sometimes this approach seems overly focused on genuine application of STEM knowledge into practice. Teaching STEM subjects using the proposed approach is not always possible and depends on the content taught. For example, there is basic knowledge in mathematics and science that is more theoretical, and does not allow practical applications being used for its explanation.

Furthermore, some STEM practices are constrained by current development of technology. There are additional obstacles, such as “difficulty of integration of the curriculum, lack of professional knowledge of teachers, lack of curricular sources” (see [6]). As pointed out in [41], “connecting ideas across disciplines is challenging when students have little or no understanding of the relevant ideas in the individual disciplines”. [27] suggested that “instead of teaching content and skills and hoping that students will see the connections to real-life application, an integrated approach seeks to locate connections between STEM subjects and provide a relevant context for learning the content”. Finally, we emphasize that there are many highly appreciated courses in higher education, dealing with topics close to some of the discussed here that do not develop a STEM approach (as a good example, see [5]).

The framework

We have adapted the model presented in [27] to fit our concepts. The proposed framework is intended for graduate students and their educators. The ideas suggested are summarized in Figure 1. Our goal is to offer a framework to help STEM educators and researchers. The figure illustrates the links between context-based and project-based learning, engineering design, scientific inquiry, technological literacy, and mathematical thinking as an integrated system. We believe that student practice, together with existing inner interdisciplinary connections, is the driving force when considering the conceptual framework for STEM learning.

Context-based and project-based learning

Context-based learning (*CBL*) was first proposed by Gilbert in [19]. Gilbert proposed four models of *context* (that appeared to be used in chemistry education, but they are relevant beyond the disciplinary content of chemistry), including: “(1) context as the direct application of concepts; (2) context as reciprocity between concepts and applications; (3) context as provided by personal and mental activity; (4) context as the social circumstances”. CBL is aimed to help learners make connections between concepts in different disciplines, and relate them to their own experiences, to the real world. Furthermore, the examples given below rely on a student-centric approach, such that students themselves explore and relate information to context, using their prior experience with computer programming. Research indicates that for STEM teaching and learning the integrator is engineering practice, and engineering design of technologies is the context. From the other hand, the engineering design or engineering practice related to relevant technologies requires the use of scientific and mathematical concepts through design justification, and the context of instruction requires solving a real-world problem or task through teamwork (see [38], [49]). Of course, there is some STEM content that cannot be placed in authentic contexts.

Project Based Learning is perceived to be a "student centred approach to learning" (see [10]. It is mainly task-oriented and the project is often set up by the teacher. The focus is on the application and assessing of previously acquired knowledge. Engineering students should be able to apply their knowledge and skills to make design solutions. These can be achieved with project based learning tasks. Principles of project based learning in common are shown in [10], and include: (1) Student's work together in groups and collaborate on project activities; (2) A real world problem that affects the life of the student's is presented for investigation; (3) Student's discuss findings and consult the teacher for guidance, input, and feedback; (4) The maturity level of student's skills determines the degree of guidance provided by the teacher.

Engineering design

Engineering design can serve as an ideal STEM content integrator (see [27]). In [18], it is stated that engineering design "provides the opportunity to locate the intersections and build connections among the STEM disciplines, which has been identified as key to subject integration". Brown et al. in [9] pointed out that: "Engineering and technology provide a context in which students can test their own developing scientific knowledge and apply it to practical problems; doing so enhances their understanding of science - and, for many, their interest in science - as they recognize the interplay among science, engineering, and technology". Engineering design and scientific research emphasize *learning by doing*. We consider this approach suitable for graduate students. Additionally, it can be outlined that not every content can be taught by design based approach.

Scientific inquiry

Scientific inquiry prepares students to ask questions, hypothesize, and conduct research, using standard scientific practices. However, this approach requires a high level of knowledge and participation on the part of teachers and students. Students can become leading factors in their learning if they have the opportunity to formulate their own questions and to make hypothesis, related to the scientific content they are researching. In our opinion, this approach is very suitable for graduate students.

Technological literacy

In the literature there are indicated two views of technology: *the engineering view*, which referred to as the instrumental perspective and *the humanities view* of technology which focuses on the human purpose of technology as a response to a specific human endeavour. According to [23], technology consists of: (1) a distinct body of knowledge; (2) an activity or a way of doing; (3) design, engineering, production, and research procedures; (4) physical tools, instruments, and artefacts; (5) organized integrated systems and organizations that are used to create, produce, and use technology. In our framework, we consider the engineering view of technology.

Mathematical thinking

According to [8], mathematical thinking involves "conjecturing, reasoning and proving, abstraction, generalisation and specialisation". As stated in [48], "in developing students thinking, the problem posed must be challenging, stimulating and within their zone of proximal development". Bowyer and Darlington in [7] outlined as "the main problems students' mathematical fluency, rather than specific content or topics". In [52] is argued that "contextual teaching can give meaning to mathematics because students want to know not only how to complete a mathematical task but also why they need to learn the mathematics. They want to know how mathematics is relevant to their lives".

Nevertheless, "incorporating STEM practices that include mathematical analysis necessary for evaluating design solutions provide the necessary rational for students to learn mathematics and see the connections between what is learned in school with what is required in STEM career skills" (see [27]). From the perspective of mathematics as a core subject in engineering education, in [46] is pointed out that "mathematics should empower students to think critically and creatively and beyond their foundational knowledge of the subject. One should be able to interpret information mathematically and utilize mathematical knowledge to address problems and arrive at solutions meaningfully". Of course, not all mathematical content can be applied to engineering design approaches.

Student practices and inner interdisciplinary connections

The interdisciplinary approach has been used in many ways and at all levels of education. Youngblood in [54] stated that "the foundation of interdisciplinary techniques will lead to a future of discovery and innovation". In [26] is pointed out that "interdisciplinary techniques are not only important for a student to learn any one single discipline or solve problem in a synthesized manner, but it also enriches a student's lifelong learning habits, academic skills, and personal growth". Kleinberg claimed that "interdisciplinary courses are viewed as necessary for attracting the best students" (see [31]). If the methodology is reduced to specialization, then students will not have a synthesis of a wide range of disciplines. To overcome this problem, Kleinberg suggests two models of interdisciplinary courses, one that covers a very broad range of disciplines, and another is a project based group that focuses more than one discipline on a specific issue. In our framework we propose to combine the two approaches in one course: *a context-best approach*, in which we offer students a wide range of interdisciplinary problems, and a *project-based approach*, in which students are given the opportunity to study more than one discipline on a specific issue.

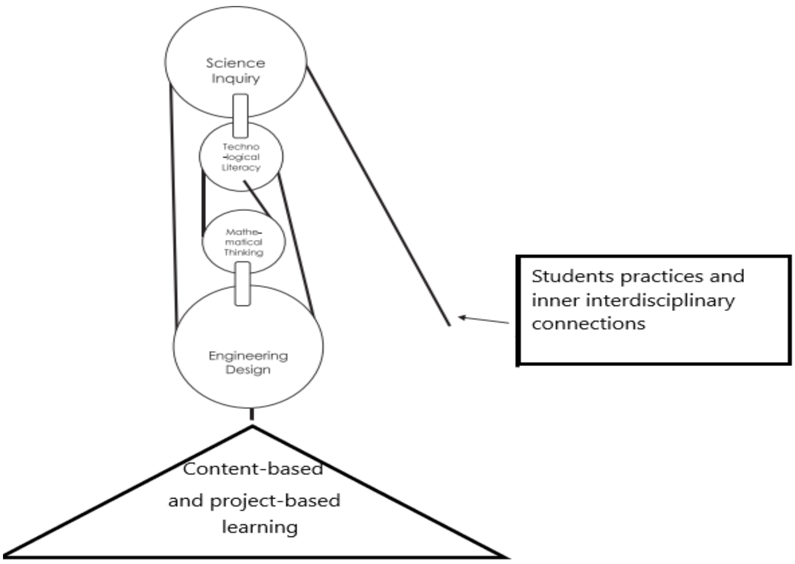

Figure. 1: *Graphic of the framework for STEM learning* (*adapted from [27]*)

Student practices are actually "science learning outcomes for students" (see [27]). According to [33], practices "build the unique set of knowledge, skills, as well as a unique language to form common practices of science and technology while investigating and solving problems". These practices provide the student with credible examples that can help illustrate interdisciplinary STEM connections. An integrated STEM approach influences the idea that STEM content should be taught along with STEM practices. Both content and practice are equally important to provide sufficient learning context. The main purpose is to help students make better decision about their future careers.

3. Software used in the educational process

Teaching graph theory, especially if the course curriculum is not entirely theoretical, can be supported by using a variety of software packages and, in some cases, as noted in [43], require a system of well-integrated media in order to deliver course materials such as images, definitions, and animations to students. The literature in this field discusses software frameworks developed by educators [1], [11], [13], [42], and [53], as well as software packages and libraries for visualization of graphs [16], [17], [44], and [42]. In addition to using a single software package to define, process and visualize a particular type of graphs, it is also possible to use a combination of different technologies. The Gephi Software (www.gephi.org), which is free and open-source, licensed under CDDL and GPLv3, is a tool with powerful graph visualization capabilities, and is a good choice for the examples discussed in this paper. It can be used to create and manipulate graphs, as well as to load graphs from XML (Extensive Mark-up Language) files, written in *gexf* format (https://gephi.org/gexf/format/). After students get familiar with the structure of this file format, they are able to create graphs using other tools, and import them to Gephi for visualization purposes. Nonetheless, the choice of this software is only a recommendation and it is technically possible to use alternative tools. For example, if the Python language (www.python.org) is used for creating and processing graph structures, the module NetworkX is a good alternative. The examples discussed below are suitable for students with at least basic knowledge in computer programming, which would allow them to implement the algorithms given in pseudo-code, as well as to conduct their own experiments. From this perspective, the Python language is convenient because it is relatively easy to learn and accessible, widely used, and creating a small Python program is usually much faster than in other programming languages. However, many programming languages can be used as the only requirement is to generate a *gexf* representation of the graphs so that they can be imported into Gephi. Hence, the pseudo-code examples in this paper are sufficiently flexible to be implemented in any popular object-oriented or scripting language (incl. Java, C#, C++, but also R, Matlab, Go, Rust, JavaScript). Indeed, the specific use of programming language can be easily adapted to the prior knowledge of the students and also allows them to individually choose the programming language to apply. One of the objectives is to encourage students to experiment and be creative by giving them the opportunity to use technologies that are convenient for them.

4. Selected examples of graphs

4.1 Example graph of integer numbers and their prime factors

Factorization of prime numbers is a very important issue. Students are interested in exploring this problem from a graph-theoretical perspective, given the fact that some prime numbers are factors of many non-prime numbers. Various questions arose, some of them are: can we build graphs representing large ranges of numbers; do their structure change; what are the limitations of the tools and technologies we use?

Providing students with the opportunity to study such examples will improve their theoretical understanding of the mathematical nature of the problem, but will also give them the opportunity to apply their analytical skills in a practical context. The procedure for generating the graph on Figure 2 is relatively simple and can be easily implemented using different programming languages. It includes elements of number theory, practical computer programming and graph theory. For a mathematics student with no extensive experience with programming languages, it may be interesting to consolidate his/her knowledge of data structures and algorithmic thinking, as well as how to visualize a solution of a purely mathematical problem.

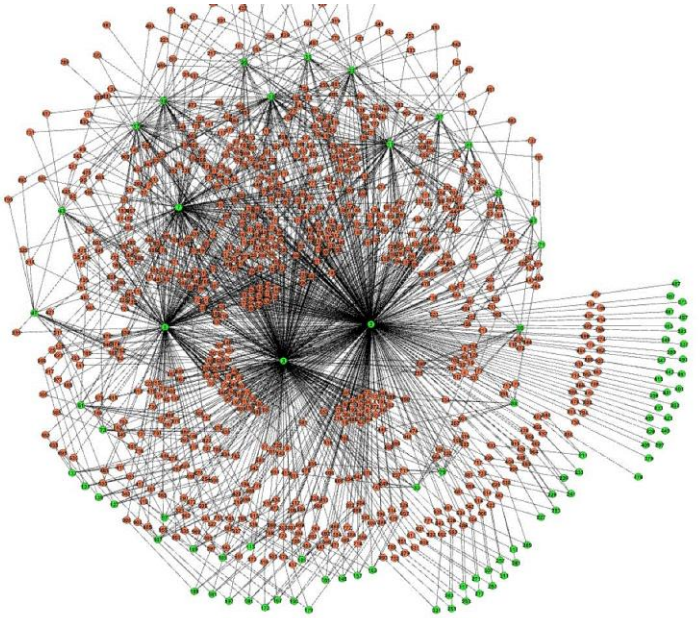

Figure 2: *Graph representation of the integers between 2 and 1000 and their prime factors (green vertices).*

Algorithm 1:

```
inputs: N //an integer (e.g. 1000)
init:
  g ← a graph with nodes {1, …, N} and empty edge set, default node colour attribute: 'brown'

function iterate(to_num):
  for i ∈ {1, …, to_num}:
    is_prime ← is_prime_number(i)  //check whether i is prime
    if is_prime is True:
      g.nodes[i].set_colour('green')  //update g: change default node colour
    else:
      factors_list ← get_prime_factors(i)  //calculate all prime factors of i (Trial division or other method)
      for factor_num ∈ factors_list:
        g.create_directed_edge(factor_num, i)  // update g: create an edge from factor_num to i

// run it:
iterate(N)
to_xml(g)  // convert the graph into XML format
```

On the other hand, a computer science student would have the opportunity to consider abstract relationships [22] between integer numbers not only as formulas on paper, but also from a more pragmatic perspective. Students from a graph theory class would easily see how this discipline is at the intersection of mathematics and computer science. This exercise can be easily extended in a way corresponding to the purpose of the course being taught and the needs and interests of students. For example, students can be asked to think about optimizing the complexity of the algorithm (for instance, the function which computes the prime factors of a given integer). Behaviour and execution can also be analysed in the case of large numbers, in terms of CPU and memory consumption. Parallelization/multithreading can be considered as well, especially for computer science students.

4.2 Example visualization of Cayley graphs and permutation groups

From one hand, the concept of groups (like Permutation groups and Symmetric groups) and their generators may seem quite abstract for some students, especially to students with dominant computer science background. On the other hand, the concept of Cayley graphs can also be difficult for them. However, graphs corresponding to small symmetric groups are easy to visualize using a simple algorithm. This example is very similar to Example 1, but emphasizes the relationships between abstract algebra and group theory with graph theory. Students are asked as an exercise to generate in an automated manner the graph representing a specific permutation group and to visualize it, since Algorithm 2 can be given in pseudo-code (as written below), with the goal of implementing it using some known programming language. Students with more advanced knowledge may be asked to try to find a solution on their own. While this solution is abstract in nature, and seemingly difficult at a first glance [48], it is not really complicated and can be implemented using a proper initialization and a single recursive function. Graph vertices correspond to a different permutation of a finite group of elements, therefore the initial vertex and the list of permutation to be performed can be passed as a parameter, and the role of the recursive function would be to sequentially apply all possible permutations, and to invoke itself until all vertices (i.e. possible permutations) are visited.

Algorithm 2:

```
inputs:
  permutations ← list of lists   //for example two permutations: [[4,1,2,3], [2,3,1,4]]
  init_node ← [‘A’, ‘B’, ‘C’, ‘D’, …]  //initial permutation
init:
  visited ← set()
  edges ← dictionary()

function recursion(node):
  for perm ∊ permutations:
    temp_node ← permute(node, perm)  //ex: BACD = permute(ABCD, [2,1,3,4])
    edges.add_edge(node, temp_node)  // add a directed edge to the edges dictionary
    if temp_node ∉ visited:
      visited.add(temp_node)
      recursion(temp_node)  // recursive call

// run it:
visited.add(init_node)
recursion(init_node)
to_xml(visited, edges)  // convert the graph into XML format
```

Interestingly, depending on the chosen permutations, graphs with the same vertices, but different edges structure can be obtained, as can be seen in Figure 3.

For example, if we want to represent the group Sym(4) as a graph, the following permutations can be applied:

- P1: ((1, 2, 3, 4), (4, 1, 2, 3)) and P2: ((1, 2, 3, 4), (2, 1, 3, 4)) to obtain the graph on the left hand side of Figure 3

- P1: ((1, 2, 3, 4), (4, 1, 2, 3)) and P3: ((1, 2, 3, 4), (2, 3, 1, 4)) which will lead to the graph on the right hand side of Figure 3

The conversion of the obtained graph into XML is simple, and working code can be given to students, so they can focus on the important part of the algorithm. It is worth to mention that this algorithm is also applicable to groups of 5, 6 or more elements, and the list of permutations to apply for the graph generation can be passed as a parameter. In fact, this example can be extended to larger groups in a similar way as described in Example 1.

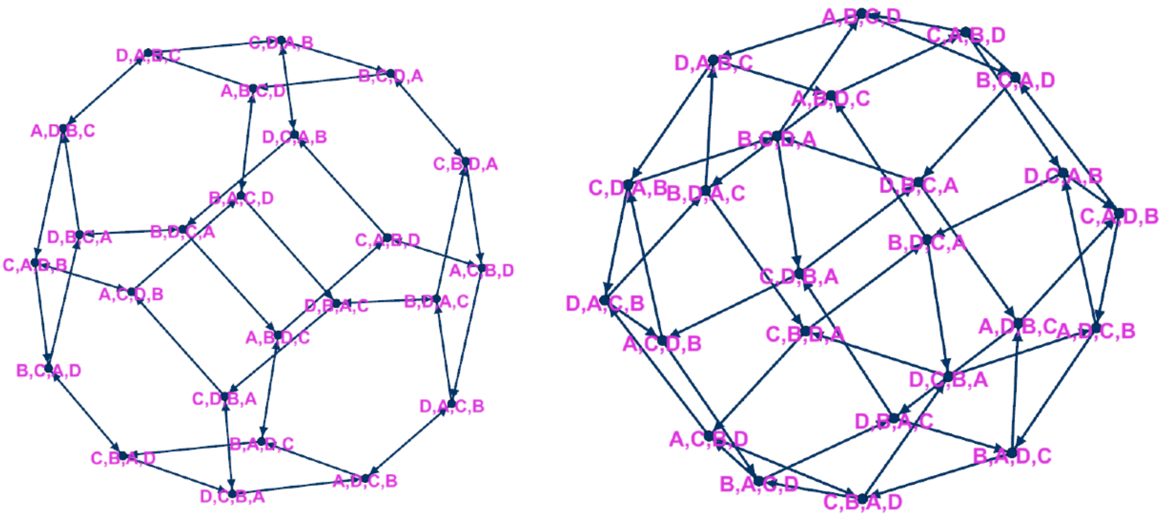

Figure 3: *Graphs corresponding to Sym(4) generated by the Algorithm 2, obtained using the following permutations: P1 and P2 (left); P1 and P3 (right)*

4.3 Examples of simulating of random networks

The applications of graph theory are not limited to applications in algebra or computer science. The behaviour of random networks, preferential attachment and scale-free behaviour [3], are at the intersection of graph theory and probability theory [34]. This relationship is emphasized by the algorithm shown below, and example of randomly generated networks. Based on this, we can draw the attention of students that graphs can represent both deterministic and non-deterministic phenomena. In particular, in the context of this example, we considered random graphs according to the Erdős–Rényi model [15], with probability of creation of a random edge controlled by the parameter *p*, and graphs exhibiting preferential attachment behaviour, similarly to the randomly growing scale-free networks studied by Barabási and Albert [3]. In the former case, the Algorithm 3 needs to be invoked with parameter *preferential = False*; and for the latter *preferential = True*, which allows the edge creation probability to depend on the current degrees of the vertices. We have also extended the preferential attachment model so that it can be controlled by the parameter *pw*, in order to amplify the preferential attachment effect.

One of the most interesting characteristics of such an exercise is that it allows students to think about random simulations and probabilistic algorithms, which is a fairly rare subject in most of the STEM specialties, with the exception of those that focus on probability, statistics and econometrics. Using such models and implementing a simulation on their basis is not always obvious, but this example exercise can be adapted as shown in the examples discussed above. Furthermore, the model and the results obtained can be considered in terms of probability distributions (Uniform, Binomial, Gaussian, and Power law), the Law of large numbers and Central limit theorem. Indeed, the two types of networks (Figure 4 left and right) exhibit the vertex degree distributions, respectively, in accordance with the Binomial law and the Power law, as depicted in Figure 5.

Algorithm 3:

```
inputs:
  p ← number ∈ [0, 1] //probability between 0 and 1
  preferential ← {True, False}  //preferential attachment or Erdős–Rényi model
  pw ← 0.0  //power parameter, equal to 0.0 by default

function random_graph(N, p, preferential, pw=0.0):
  vertices ← {1, …, N}
  edges ← set()
  deg ← [0, 0, …, 0] //vertex degrees initialization - N zeros

  for i ∈ {1, …, N}:
    for j ∈ {1, …, i-1}:
      u ← U([0, 1]) //uniformly distributed random number between 0 and 1
      if preferential = = True:
        p ← max(deg_i, deg_j)^(1+pw) / (1 + ∑_{k=0}^{i} deg_k)
        if deg_j == 0:
          edges.add_edge(j, r) where r is a randomly selected vertex
          deg_j ++; deg_r ++
      if u > 1 − p :
        deg_i ++; deg_j ++
        edges.add_edge(i, j)
  return g, deg

// run it:
g, deg = random_graph(250, 0.2, False, 0.0)
to_xml(g)  // convert the graph into XML format
plot(histogram(deg))
```

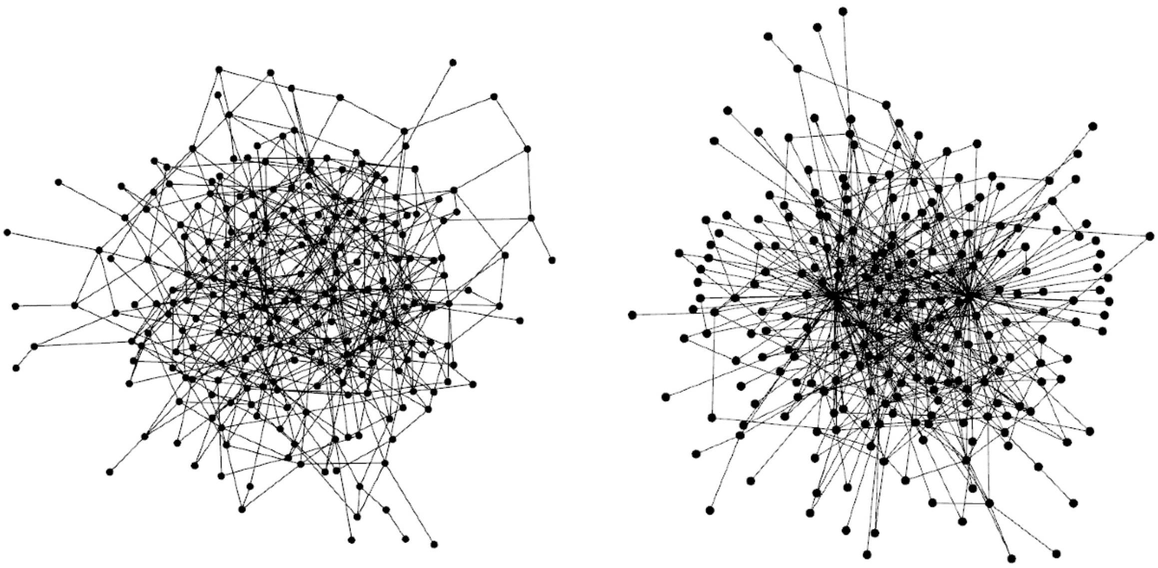

Figure 4: *Random graphs generated by the Algorithm 3: an Erdős–Rényi graph (left) and a graph exhibiting preferential attachment (right)*

This algorithm and exercise allow to perform various experiments in different directions, relevant to the education process [52], including: potential computation optimizations (for example, regarding the generation of random numbers, or using vectorization techniques in languages like R and Python); analysis of the algorithmic complexity; changing the method of simulating preferential attachment; choosing efficient data structures during the network generation process; performing analysis of the obtained probability distributions (the asymptotic behaviour, and Kolmogorov-Smirnov tests [25]); more detailed analysis of the graph theoretical properties of the obtained networks; and also trying to simulate clustering effects. This exercise can also be done as a homework assignment [10] if appropriate for the level of the students.

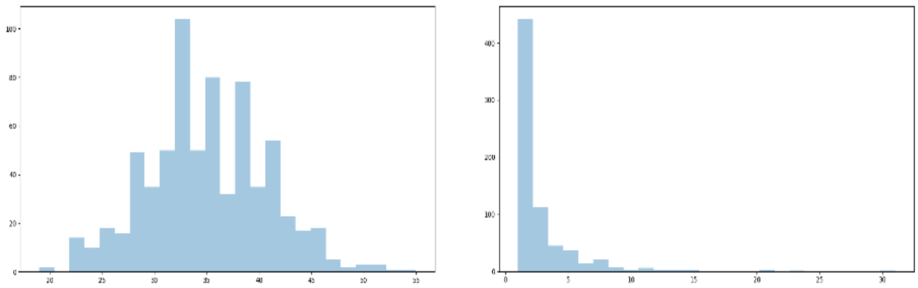

Figure 5: *Vertex degree distributions for random graphs generated by the Algorithm 3: An Erdős–Rényi graph (left) and a graph obtained by simulating preferential attachment behaviour (right)*

4.4 Example graph obtained from real-world data

Unlike the previous example, where networks are simulated using pre-specified rules and distributions, the example below is somewhat opposite – the structure of existing real-world relationships is inferred using empirical data. This example is more complicated compared to the above, not only because of the potentially complex structure of the obtained graphs, but also due to the fact that it requires additional work for the raw data pre-processing, which supposes some domain-specific knowledge of finance. However, the collection and pre-processing of financial series (cleaning, calculation of log-returns) can be done by the educator, so students can focus on the main subject. Similarly to many complex systems, financial markets exhibit complex time-varying relationships, regardless of whether we consider stocks, derivatives, foreign exchange, or other assets.

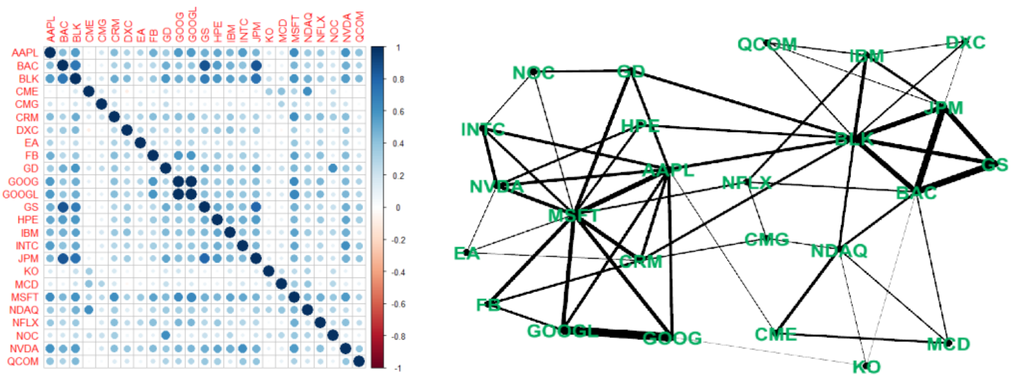

Figure 6: *A heat-map of a correlation matrix and a graph obtained by Algorithm 4 (edge size in function of weights i.e. the correlation coefficients)*

For educational purposes, we only consider correlations of stock returns of a small sample of popular highly capitalized companies. The most straightforward way to show these relationships is to visualize their correlation matrix, as depicted on Figure 6. Nonetheless, graphs can be a useful tool for both the visualization and better understanding of the underlying relationships. By filtering out weak correlations close to 0, or selecting only the strongest connections for each node, we can obtain a meaningful network.

In Figure 6, it can be seen a graph of 25 vertices, corresponding to the stocks return series, represented by the correlation matrix heat map. After filtering out the edges corresponding to small correlation coefficients using Algorithm 4, it is obtained a structure where clusters can be found (technology companies, financial institutions), and vertices with higher degrees corresponding to companies that can be considered more influential from the perspective of this network analysis. It should be noted that larger and more complex networks can be generated using exactly the same approach and algorithm.

Algorithm 4:

```
inputs:
  S ← {s_1, … , s_n} //time series of closing prices corresponding to n different stocks
  k ← integer //indicates to retain for each vertex the k edges corresponding to the highest correlations

function generate_graph(S, k):
  R ← log-returns(S) //obviously, over the same time periods
  C ← correl(R) //compute the n x n correlation matrix of the return series
  V ← {stock_1, … , stock_n} //vertex labels correspond to stock tickers
  edges ← dictionary() //keys of the form {from, to}, values are floating numbers representing weights

  for i ∊ {1, …, n}:
    c_{i,:} ← |(c_{i,:})| //a row vector
    sorted ← sort((c_{i,0}, i, 0), …, (c_{i,n}, i, n)) //sort the triplets according to the correlation coefficients c
    sorted ← sorted_{1:k}  //get the highest k correlation coefficients (and the corresponding row/col indexes)

    for {c, ii, jj} ∊ sorted: //iterate over the triplets (c, row index, column index)
      if edges.contains( {ii, jj} ) or edges.contains( {jj, ii} ):
        continue //because we are generating an undirected graph and the correlation matrix is symmetric
      else:
        edges.add_edge(key ← {ii, jj}, value ← c)  //the dictionary value c would be the edge weight

  return graph(V, edges)

// run it:
g = generate_graph({s_1, … , s_25}, k ← 3):
to_xml(g)  // convert the graph into XML format
```

We suggest giving this to students as project-type assignment [10] along with providing appropriate guidance and advices. For students with good computer science background, there is a plenty of technical issues to consider [27], including efficiency of the code, appropriate data structures and collections in function of the programming language used, and even design patterns and unit tests if the project assignment is extended with additional requirements (e.g. larger networks, comparisons of different time periods, computations for more sophisticated network analysis). For students with more theoretical background, such a project would be a good opportunity to explore real-world data and to produce interpretable results (see also *Scientific inquiry* in Sect. 2). The project can also be extended in such a way that it is of interest to students studying mathematics, statistics or data science

– for example, to perform more sophisticated filtering procedures; to use causality tests instead of correlation; or to try to detect clusters in larger networks.

5. Study

The examples discussed in the previous section were given to a class of 12 graduate-level students as a sequence of homework assignments, in a project-based setting (see Sect. 2 and [10]). All students had a prior background in a technical discipline, such as computer science, engineering, applied mathematics or telecommunications. In addition to lectures, which are more theoretically oriented and discuss core graph-theoretical concepts, during our weekly lab sessions we discussed potential difficulties related to the implementation of these examples and best practices for implementing programming solutions. All students in the group are pursuing careers in industries involving programming and a good knowledge of contemporary technologies. According to the survey outlined below, they all managed to complete their project assignments successfully, and were happy with the project-based approach.

As a source of potential difficulties, we identified that students in the group studied in different bachelor's programmes, therefore, have different levels of experience with specific technologies. This is one of the reasons to allow everyone to use any applicable programming language without restrictions. Moreover, we think it would be good to encourage students from different undergraduate programmes to work in a group, so they can exchange knowledge, help each other and discuss different aspects of practical solutions. The survey includes questions about how students perceive the combination of various disciplines within a single project/task [26], [27], and [33] and whether they find the combination of mathematics, computer science, theoretical knowledge and specific technologies useful and helpful for their future career paths. Although we received 8 responses (out of 12), which cannot be considered statistically representative for a large population, the results indicate that students are satisfied with the project-based interdisciplinary approach and find it helpful. Furthermore, the answers received show that overall student engagement has increased while working on the projects. Affirmative answers to Q3 show the correctness of our beliefs that contextual teaching can give *meaning to mathematics* [52]. Students pointed out that learning through exercises, homework and projects shows them how mathematics is exciting and relevant to improving their skills. On the other hand, contextual and project-based teaching consolidate their technological literacy. (All the students indicated this). Relating Q9, students stated that "project assignments are very useful for us for the job interview, especially in the final phases". Moreover, also when answered the question Q15, a student said that "as to me, this is the best way for learning"; another shared that: "when a person is faced with a real problem, he must find a solution"; there is even an answer "I learn better, when the study material "goes through my hands"". This is a confirmation of the fact mentioned above that engineering design and scientific research emphasize *learning by doing*.

Survey questions:

Q1: What is your Bachelor's degree in?
Q2: Taking into account your undergraduate education, do you agree that, it would be useful to include in the educational process more interdisciplinary exercises and project assignments including knowledge from more than one subject?
Q3: Do you agree that exercises, homework and projects, combining mathematics, programming, and sometimes other disciplines, would help you to consolidate your math knowledge?
Q4: Do you agree that exercises, homework and projects, combining mathematics, programming, and sometimes other disciplines, would help you to consolidate your knowledge on computer science?
Q5: Do you think that these project-based assignments are helpful to better understand what (and why) we are discussing during the lectures?
Q6: Do you agree that learning Graph theory along with a mathematically-inspired problem, and implementing a computer program, is a better option than studying only pure Graph theory, outside the context of other disciplines?
Q7: During the lab sessions, we are discussing the solutions and difficulties you face in completing the homework projects. Do you think it would be good if we allocate more time for:

a) The underlying theory?
b) The data structures and algorithms (to be) used?
c) Optimization and efficient execution of the programs?
d) Discussions of other similar problems?
e) Other (please specify)?

Q8: Do you think that visualizing the obtained results will help you better understand what the real problem is and interpret the solutions obtained?
Q9: Do you agree that interdisciplinary exercises and projects, would be useful for you in the future, from the perspective of your career path?
Q10: Do you find the project assignments interesting? Why?
Q11: Do you think that the practical implementation of your solutions (incl. our on-line discussions and quick research on the Internet), would be useful for you from a professional perspective? Please motivate your answers.
Q12: Do you think you have improved your research capabilities to some extent when it comes to finding practical solutions to real-world problems? Why?
Q13: Are the methods/technologies used within this course relevant from a professional perspective?
Q14: Did you find the homework projects difficult? Why - please, motivate your answers?
Q15: Do you find the project-based approach used in the assignments in this course suitable and useful for you, from a general perspective? Why - please, motivate your answers?

Responses (by question):

Table 1: Answers to Question 1:

Q1	Computer science	Mathematics	Telecommunications	Electronic engineering	Other
count	3	2	1	1	1

Table 2: Answers to Question 2:

Q2	Yes	No
count	8	0

Table 3: Answers to Question 7:

Q7	count
the theory	1
data structures and algorithms	7
optimisation and efficiency	4
discussion around similar problems	3
other	1

Table 4: Answers to Questions 3 to 6, and 8 to 15:

Question	No	Rather no	Neutral	Rather yes	Yes
Q3	0	0	1	3	4
Q4	0	0	0	2	6
Q5	0	0	0	0	8
Q6	0	0	1	2	5
Q8	0	0	0	3	5
Q9	0	0	0	2	6
Q10	0	0	0	1	7
Q11	0	0	1	2	5
Q12	0	0	2	3	3
Q13	0	0	0	2	6
Q14	1	2	3	1	1
Q15	0	0	0	0	8

Selected answers to some questions:

Q3: In my opinion, the combination is definitely much better. It becomes more interesting when the assignment combines different fields.

Q9: It is a fact that project assignments are very useful for us for the job interview, especially in the final phases.

Q10: I find the material taught interesting, because we are actually studying different types of problems that we solve through programming code implementation. It forces us think logically and search for different solutions.

Q11: Rather yes, because when we work on a real (world) task, it is useful to search for more information from various sources.

Q13: At the present moment, I do not actually need to use graph theory and related technologies, but I think that in the future, due to the development of machine learning, big data, and so on, this course would be useful to me.
Q13: I am learning new things about the programming language (which I am using), and improving my coding skills in what concerns the algorithms and methods.

Q14: At the moment, the assignment task was very interesting, I did not encounter many difficulties, but found new things that are useful to me.
Q14: Rather yes, because until now and during my bachelor's degree related to electronics, I have been using the C programming language for low-level programming.
Q14: During my undergraduate studies, I was not very interested in mathematics, but now, step by step, I realize how interesting it can be. The difficulty for me was that I needed to search by myself for more information, even for simple mathematical concepts.

Q15: Yes, because when a person is faced with a real problem, he must find a solution. I think that this approach is better than just listening to theoretical lectures.
Q15: This approach is suitable to me, as it includes two things that interest me: informatics (coding) and mathematics.
Q15: Without projects, I would not be spending time on reading theory - there are no direct benefits to me. This way (with project assignments), I have a motivation to understand how to solve a specific problem. In other words, without projects, I would simply not be interested.
Q15: As to me, this is the best way for learning. Studying only theory and preparing for theoretical exams would be much more difficult and time consuming for me. On the other hand, when we consider examples, including doing some research on the Internet, we are successfully combining things that are interesting, useful and fundamental, namely algorithms, data structures, math and programming.
Q15: Definitely yes, because working on a fairly big task/exercise and solving the problem gradually, including searching for more information, is a much better approach than solving a big number of small, simple and "artificial" exercises.
Q15: I learn better, when the study material "goes through my hands", rather than studying only pure theory.

The answers show that students strongly agree that the examples discussed in Section 4 improved their understanding of the material being taught, they find it interesting and their engagement increased (Q2, Q5, Q10, and Q15). They also tend to agree that the assignments are professionally relevant (Q9 and Q11), and therefore suitable for their future careers. Answers to Q14 indicate that the perception whether the assignments are difficult varies across students, most likely because of their different level of preparation, different undergraduate programmes, and personal preferences towards what skills are best for them. As mentioned, we have been regularly discussing with the students the suitable solution approaches and the difficulties encountered. Considering the discussions during lab session, as well as the answers in Table 3, it becomes clear that most of the students want to improve their knowledge about data structures in programming, the algorithms used, and the efficiency of the implemented solutions. Some students also stated that they would like to learn more about similar practical problems, which shows that they want to deepen their knowledge, and further expand their skill set. Our impression is that these answers are a consequence of the fact that our students are already looking forward to practical needs in their professional career (some of them looking to pursue a PhD degree). So, while studying this material, they are aware of the opportunities and challenges related to their future careers and they appreciated the assignments, because they found them relevant and applicable. In particular, group work and regular discussions with the teaching assistant are useful and help them to overcome potential difficulties. We believe that the example assignments discussed in this paper are applicable to a wider audience, as they can be adapted from both a math and code perspective, in a way that they remain interesting and useful for different groups of students in late undergraduate or master programmes.

6. Conclusions

STEM education allows students "to handle existing problems with an interdisciplinary approach and thus, enables them to access the knowledge and skills expected" [4]. Thus, this educational approach is very important in terms of teaching students, allowing them to learn and then specialize in the various fields in which they will work. An interdisciplinary approach "offers the advantage of presenting the knowledge utilization as integrated rather than discipline-specific when solving real-world problems, it is yet to be established that this is beneficial to all students", however "there is an evidence that interdisciplinary approaches can support the learning of low achieving students" [35]. On the one hand, it is challenging to conduct research in STEM education as this is a fairly new field, and it is not fully established what best practice is. On the other hand, there is still "no clear understanding of the role of individual subjects within integrated STEM" [35]. All this shows that applying interdisciplinary approach to STEM teaching as an everyday teaching practice remains difficult. However, it is possible to formulate and develop suitable exercises and project assignments within the intersection of mathematics, graph theory and computer science, in order to enhance their scientific research interest and mathematical thinking. In this paper we have shared our ideas and experience on how to implement an interdisciplinary, context-based educational approach in a project-based setting in teaching graph theory. We have suggested four examples from our practice that we find useful for the teachers in their future implementation of this approach, especially from the perspective of math, programming and graph theory. These examples are especially relevant to students that are looking to improve their programming skills. As our practice and the answers given by students in the survey show, there is an increased interest of students in the study of graph theory, as well as in participation in discussions for solving real-world problems. This motivates us to continue our research on the implementation and extension of these approaches in teaching graph theory.

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